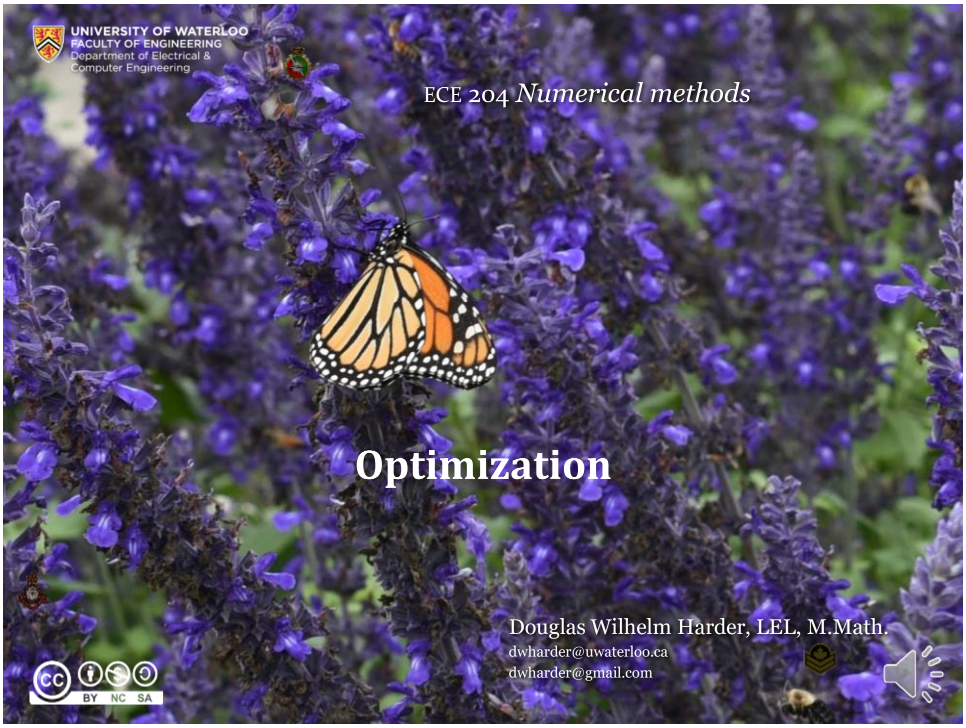

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Introduction

- In this topic, we will
  - Describe what will be covered in the next section
    - Optimization
  - Give definitions of various terms
  - Discuss constrained optimization
  - Describe some issues with optimization versus root finding
  - Describe the two major topics in this section

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Optimization

- Optimization involves the process of,
  given a function $f(x)$, find an argument
  - x_{max} such that $f(x_{max}) \geq f(x)$ for all x of interest
  - x_{min} such that $f(x_{min}) \leq f(x)$ for all x of interest
- The first process is called *maximization* and the second *minimization*
  - Solutions to the first are *maxima* while solutions to the second are *minima*, together known as *extrema*
  - As maximizing $f(x)$ is equivalent to minimizing $-f(x)$,
    we will focus entirely on minimization processes in this section

Optimization

- A *global minima* is a point x_{min} such that $f(x_{min}) \leq f(x)$ for all possible arguments of x
  - This is actually a very difficult problem
- A *local minima* is a point x_{min} such that $f(x_{min}) \leq f(x)$ for all x sufficiently close to x_{min}
  - More rigorously, there exists an $\varepsilon > 0$ such that $f(x_{min}) \leq f(x)$ for all x such that $|x - x_{min}| < \varepsilon$

Optimization

- As with root-finding, we will distinguish these two cases:
  - Optimizing a real-valued function of a real variable
    - One dimensional
  - Optimizing a real-valued function of a vector variable
    - Higher dimensional

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Constrained versus unconstrained

- In some cases, constraints may be put on the possible solutions
  - Minimize $f(\mathbf{x})$ subject to $\mathbf{x}$ satisfying these conditions: ....

- For example, find an n-dimensional vector $\mathbf{x}$ that maximizes

  $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ where $\mathbf{c}$ is an n-dimensional vector

  where

  $\mathbf{x} \geq \mathbf{0}$ or all entries of $\mathbf{x}$ are non-negative

  $A\mathbf{x} \leq \mathbf{b}$ for a given matrix A and a vector $\mathbf{b}$

  - There are excellent upper-year undergraduate courses that focus specifically on such types of optimization

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Issues with optimization

- When you find a root, it can usually be found to significant accuracy
  - For example, finding a root of sin(*x*) can be done close to sixteen significant digits
  - This is due to the ability to represent numbers as small as 10^{-300}

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Issues with optimization

- If we are finding an extrema with function evaluations, there may be many numbers that map to the same floating-point value

$$f(x) \approx f(x_{\min}) + \frac{1}{2} f^{(2)}(x_{\min})(x - x_{\min})^2$$

- If the error is too small, less than $f(x_{\min})\, 2^{-52}$, then $f(x) = f(x_{\min})$

$$\frac{1}{2}\left|f^{(2)}(x_{\min})\right|(x - x_{\min})^2 < \left|f(x_{\min})\right| 2^{-52}$$

$$(x - x_{\min})^2 < \frac{2\left|f(x_{\min})\right|}{\left|f^{(2)}(x_{\min})\right|} 2^{-52}$$

$$\left|x - x_{\min}\right| < \sqrt{\frac{2\left|f(x_{\min})\right|}{\left|f^{(2)}(x_{\min})\right|}}\, 2^{-26}$$

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Optimization

Issues with optimization

- For example, in finding the minimum of sin(x), we have that a point will appear to be a minimum if

$$\left|x - x_{\min}\right| < \sqrt{2} \cdot 2^{-26} \approx 2.107 \times 10^{-8}$$

  – Thus, we can have at most approximately eight digits of accuracy

```
>> format long
>> sin( -pi/2 )
    ans = -1
>> sin( -pi/2 + 1e-8 )
    ans = -1
>> sin( -pi/2 - 1e-8 )
    ans = -1
>> sin( -pi/2 - 1e-7 )
    ans = -0.999999999999995
```

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Optimization

Issues with optimization

- Essentially, as far as the computer is concerned, sin(x) has a minimum at every value of x on the interval

[–1.570796337, –1.570796317]

  – The correct answer is –1.57079632679489 7
  – Thus, you cannot have more than eight significant digits

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Looking ahead

- In this section, we will look at:
  - Unconstrained optimization finding local minima of a real-valued function of a real variable
  - Unconstrained optimization finding local minima of a real-valued function of a vector variable

Summary

- Following this topic, you now
  - Have an overview of the ideas to be covered in this section
  - Are aware of some of the ideas behind optimization
  - Understand that we will first optimize functions of a real variable and then go on to functions of a vector variable
  - Are aware that optimization may not be as accurate as root finding

References

[1] https://en.wikipedia.org/wiki/Mathematical_optimization

Acknowledgments

None so far.

Optimization

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

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Optimization

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